Anosov diffeomorphisms on infra–nilmanifolds

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After graduating from the secondary school Sancta Maria Instuut in Aarschot in 2001, I studied mathematics at the Catholic University of Leuven, obtaining a licentiate degree in mathematics in June 2005. From October 2005 till September 2009, I was a Research Fellow of the F.W.O.-Flanders. Under guidance of Prof. Dr. Karel Dekimpe, I prepared my PhD thesis, which was defended on September 23, 2009.

The research to Anosov diffeomorphisms started in 1967, when S. Smale constructed some examples of Anosov diffeomorphisms on nilmanifolds and raised the problem of classifying all Anosov diffeomorphisms on compact manifolds ([4]). Up till now, the only known examples of closed manifolds admitting an Anosov diffeomorphism are infra–nilmanifolds, and it has been conjectured that this is the only class of manifolds in which one can expect to find Anosov diffeomorphisms. Nevertheless, the knowledge about Anosov diffeomorphisms on infra–nilmanifolds is, except for the case of nilmanifolds and flat manifolds, almost non-existing. Since the infra–nilmanifolds are an immediate generalization of the flat Riemannian manifolds - the universal cover of an infra–nilmanifold is a simply connected, connected nilpotent Lie group instead of an abelian one - I tried to generalize Porteous' result for flat manifolds to (certain classes of) infra–nilmanifolds.

Main concepts

Let $f: M \to M$ be a C^1-diffeomorphism. We say that f is an Anosov diffeomorphism if the tangent bundle splits up as follows:

$$TM = E^s \oplus E^u$$

such that for some Riemannian metric $\| \ \|$ and real constants $c > 0$ and $0 < \lambda < 1$ we have $\forall n > 0$:

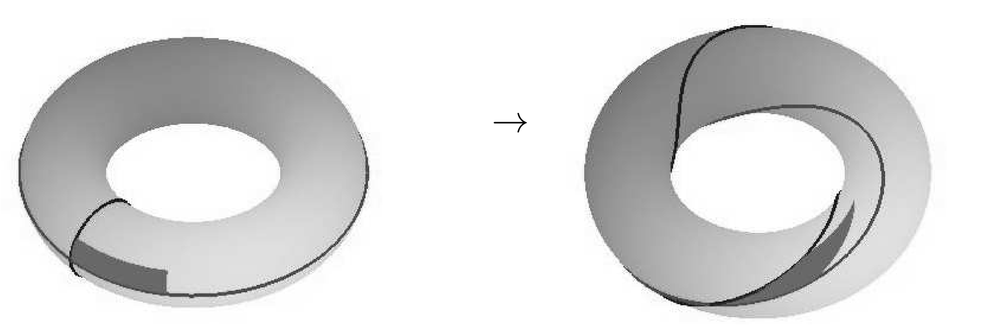
$$\|df^n(v)\| \leq c\lambda^n \|v\| \text{ for } v \in E^s \text{ and}$$
$$\|df^n(v)\| \geq c\lambda^{-n} \|v\| \text{ for } v \in E^u.$$

Let L be a connected, simply connected nilpotent Lie group, and E an almost-Bieberbach group (AB-group). Then $M = E\backslash L$ is an infra-nilmanifold with fundamental group E. If moreover $E \subseteq L$, then M is called a nilmanifold.

Main question: When does an infra–nilmanifold admit an Anosov diffeomorphism?

Example

The induced map $f: \mathbb{T}^2 \to \mathbb{T}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$



is an example of an Anosov diffeomorphism.

Special cases / Known results

Theorem [Manning, 1974, [2]] Any Anosov diffeomorphism of an infra-nilmanifold is topologically conjugate to a hyperbolic automorphism of that infra-nilmanifold.

⇝ Main question: When does an infra-nilmanifold admit a hyperbolic infra-nilmanifold automorphism?

Theorem [Porteous, 1972, [3]] A flat manifold with rational holonomy representation $\varphi: F \to \mathrm{GL}(n, \mathbb{Q})$ admits an Anosov diffeomorphism if and only if each $\mathbb{Q}$–irreducible component of φ which is of multiplicity 1, is reducible over $\mathbb{R}$.

Theorem [Dani, 1974, [1]] A nilmanifold $N\backslash L$ with L free c-step nilpotent on n generators admits an Anosov diffeomorphism if and only if $n > c$.

Rational holonomy representation

Let E be an AB–group with corresponding infra-nilmanifold $M = E\backslash L$. We know by the first Bieberbach theorem that $N = E \cap L$ is a uniform lattice of L, and that E/N is finite. This gives rise to a short exact sequence

$$1 \to N \to \pi_1(M) = E \to F \to 1,$$

in which F is called the holonomy group of $E\backslash L$.

Denote with $N_{\mathbb{Q}}$ the radicable hull or rational Mal'cev completion of N, such that $\mathfrak{n}_{\mathbb{Q}}$ is the rational Lie algebra associated to $N_{\mathbb{Q}}$. The essential extension induces the following commutative diagram of groups:

$$\begin{array}{ccccccccc} 1 & \to & N & \to & E & \to & F & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N_{\mathbb{Q}} & \to & E_{\mathbb{Q}} = N_{\mathbb{Q}} \rtimes F & \to & F & \to & 1 \end{array}$$

We can fix a splitting morphism $s: F \to E_{\mathbb{Q}}$ of the bottom extension, leading to the rational holonomy representation

$$\varphi: F \to \mathrm{Aut}(N_{\mathbb{Q}}) = \mathrm{Aut}(\mathfrak{n}_{\mathbb{Q}}) : f \to \varphi(f)$$

with

$$\varphi(f): N_{\mathbb{Q}} \to N_{\mathbb{Q}} : n \mapsto s(f) n s(f)^{-1}.$$

A first algebraic characterization

Theorem Let M be an infra-nilmanifold with associated rational holonomy representation $\varphi: F \to \mathrm{Aut}(N_{\mathbb{Q}})$. Then

M admits an Anosov diffeomorphism

$\Updownarrow$

there exists an integer-like hyperbolic $\psi \in \mathrm{Aut}(N_{\mathbb{Q}})$ that commutes with any element of $\varphi(F)$.

⇝ Main question: When does there exist an integer-like, hyperbolic automorphism $\psi \in \mathrm{Aut}(N_{\mathbb{Q}}) = \mathrm{Aut}(\mathfrak{n}_{\mathbb{Q}})$ that commutes with every element of $\varphi(F)$?

Infra–nilmanifolds modeled on free nilpotent Lie groups

From now on we suppose that G is a free c-step nilpotent Lie group (on k generators).
A rational holonomy representation

$$\varphi: F \to \mathrm{Aut}(\mathfrak{n}_{\mathbb{Q}})$$

induces an abelianized rational holonomy representation:

$$\bar{\varphi}: F \to \mathrm{Aut}\left(\frac{\mathfrak{n}_{\mathbb{Q}}}{[\mathfrak{n}_{\mathbb{Q}}, \mathfrak{n}_{\mathbb{Q}}]}\right) \cong \mathrm{GL}(k, \mathbb{Q})$$

General criterion in the free case:

Theorem Let M be an infra-nilmanifold modeled on a free c–step nilpotent Lie group and with abelianized rational holonomy representation $\bar{\varphi}: F \to \mathrm{Aut}\left(\frac{\mathfrak{n}_{\mathbb{Q}}}{[\mathfrak{n}_{\mathbb{Q}}, \mathfrak{n}_{\mathbb{Q}}]}\right)$. Then

M admits an Anosov diffeomorphism

$\Updownarrow$

There exists an integer-like, c-hyperbolic automorphism $\bar{\psi} \in \mathrm{Aut}\left(\frac{\mathfrak{n}_{\mathbb{Q}}}{[\mathfrak{n}_{\mathbb{Q}}, \mathfrak{n}_{\mathbb{Q}}]}\right)$ that commutes with any element of $\bar{\varphi}(F)$.

Translation into $\mathbb{Q}$-irreducible components

Theorem Let $T: F \to \mathrm{GL}(n, \mathbb{Q})$ be a representation of a finite group F. Then:

$\exists C \in \mathrm{GL}(n, \mathbb{Q})$ integer-like and c-hyperbolic, that commutes with every element of $\mathrm{Im}(T)$

$\Updownarrow$

For every $\mathbb{Q}$-irreducible component $T_i: F \to \mathrm{GL}(n_i, \mathbb{Q})$ of T that occurs with multiplicity $m_i \leq c$ we have:

$\exists C_i \in \mathrm{GL}(m_i n_i)$ integer-like and c-hyperbolic, that commutes with $\mathrm{Im}(T_i \oplus \cdots \oplus T_i)$ (m_i times).

Using (the proof of) Dirichlet's Unit Theorem, we find the following important theorem:

Theorem Let $\mathbb{Q} \subseteq K$ be a normal, totally imaginary field extension with $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \sigma_2, \ldots, \sigma_t, \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_t\}$. Then $\exists \mu \in \mathcal{U}_K$ such that

$$\forall k \in \{1, \ldots, t-1\}, \forall i_1, \ldots, i_k \in \{1, \ldots, t\} :$$

$$|\sigma_{i_1}(\mu)\sigma_{i_2}(\mu) \cdots \sigma_{i_k}(\mu)| \neq 1.$$

Classification for infra-nilmanifolds with abelian holonomy group

Theorem Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group, with abelian holonomy group F and abelianized rational holonomy representation $\bar{\varphi}: F \to \mathrm{Aut}(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]})$. Then

M admits an Anosov diffeomorphism

$\Updownarrow$

Each $\mathbb{Q}$-irreducible component $\bar{\varphi}_i$ of $\bar{\varphi}$ that occurs with multiplicity m, splits in more then $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

General holonomy group

Theorem Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group on n generators, with $n \leq 2c + 1$. If M admits an Anosov diffeomorphism, then M has an abelian holonomy group.

Corollary Let M be an infra-nilmanifold modeled on a free c-step nilpotent Lie group on n generators, with $n \leq 2c+1$, and with associated abelianized rational holonomy representation $\bar{\varphi}: F \to \mathrm{Aut}(\frac{N_{\mathbb{Q}}}{[N_{\mathbb{Q}}, N_{\mathbb{Q}}]})$. Then

M admits an Anosov diffeomorphism

$\Updownarrow$

F is abelian, and each $\mathbb{Q}$-irreducible component $\bar{\varphi}_i$ of $\bar{\varphi}$ that occurs with multiplicity m splits in more then $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

References

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[4] Smale, S. *Differentiable dynamical systems*. Bull. Amer. Math. Soc., 1967, 73, pp. 747–817.