Cohomology of polycyclic-by-finite groups and aspherical manifolds using polynomial crystallographic actions

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Cohomology, a friend once said to me, is a wild area. For instance, determining the Betti numbers for a class of Lie algebras inevitably leads to combinatorial difficulties. The situation for groups or topological spaces is even worse, as their cohomology definitions use complexes which are incredibly hard to control. Things start to brighten up a bit when studying the connections between the cohomology of a group, of its Eilenberg-Mac Lane space, and of its Lie algebra counterpart. In particular, for finitely generated torsion free nilpotent groups — so-called $\mathcal{T}$-groups — the interplay between these types of cohomology has led to nice results. In this thesis we attempt to provide some insight in the way group cohomology is built up, and exploit the connection between cohomology of groups, manifolds and Lie algebras to develop tools for an explicit description of group cohomology, in particular for the class of polycyclic-by-finite groups and with coefficients in a finite dimensional real vector space equipped with any module structure.

Cohomology of $\mathcal{T}$-groups

In 1954, Nomizu realized that the correspondence between $\mathcal{T}$-groups, finitely generated simply connected nilpotent Lie groups and finite dimensional nilpotent Lie algebras implies an equally close connection between their cohomologies [3]. Both Lambe [2] and Fried, Goldman and Hirsch [1] used this result to compute the cohomology of a $\mathcal{T}$-group with trivial coefficients in $\mathbb{R}$. A particular type of representation for a finitely generated simply connected nilpotent Lie group allowed them to transfer Nomizu's Lie group structure to an equivalent structure on a more manageable space, typically $\mathbb{R}^n$, and compute cohomology in this other realm. Where Lambe used Mal'cev coordinates, Fried, Goldman and Hirsch based their result on a study of simply transitive affine actions for finitely generated nilpotent Lie groups. Unfortunately, not every such Lie group admits a simply transitive affine action, and moreover, there is no criterion to decide whether such an action exists. However, the idea of using some kind of representation to translate the group structure of a $\mathcal{T}$-group to an action on $\mathbb{R}^n$ turns out to be very valuable.

In contrast to affine crystallographic actions for $\mathcal{T}$-groups, every $\mathcal{T}$-group does have a polynomial crystallographic action. As Fried, Goldman and Hirsch did with affine actions, we use these polynomial actions to translate Nomizu's connection between the real cohomology of a $\mathcal{T}$-group and the corresponding Lie algebra cohomology with real coefficients. This translation leads to the following

Theorem. *Let N be a $\mathcal{T}$-group and $\rho : N \to \mathcal{P}(\mathbb{R}^n)$ a polynomial crystallographic action of bounded degree. Then the ring $H^*(N, \mathbb{R})$ is isomorphic to the cohomology of the finite dimensional complex of polynomial differential forms ω on $\mathbb{R}^n$ satisfying*

$$\rho(n)^*\omega = \omega \qquad \text{for all } n \in N.$$

This invariance condition, based on a polynomial crystallographic action, actually yields an explicit formula for the invariant polynomial differential forms, thus leading to an algorithm for the computation of the cohomology ring of a $\mathcal{T}$-group with coefficients in $\mathbb{R}$.

For Lie algebras $\mathfrak{g}$ over $\mathbb{R}$, cohomology with coefficients in a finite dimensional real $\mathfrak{g}$-module, in particular the adjoint module, is equally important as trivial real coefficients. As a group cohomology pendant, we develop a computational approach to the cohomology of a $\mathcal{T}$-group with coefficients in a finite dimensional real vector space $\mathbb{R}^k$, equipped with any module structure. As for cohomology with coefficients in $\mathbb{R}$, the group cohomology with coefficients in $\mathbb{R}^k$ corresponds to the cohomology of a finite dimensional complex of invariant tuples of forms.

Theorem. *Let N be a $\mathcal{T}$-group and $\rho : N \to \mathcal{P}(\mathbb{R}^n)$ a polynomial crystallographic action of bounded degree. Suppose $\varphi : N \to GL(k, \mathbb{R})$ gives $\mathbb{R}^k$ a N-module structure, and let $(\mathbb{R}^k)_u$ be the maximal unipotent N-submodule of $\mathbb{R}^k$, say of dimension l and with module structure $\varphi_u : N \to Tr_1(l, \mathbb{R})$. Then $H^*(N, \mathbb{R}^k)$ is isomorphic to the cohomology of the finite dimensional cochain complex of l-tuples ω of polynomial differential forms on $\mathbb{R}^n$ satisfying*

$$\varphi_u(n) \cdot \rho(n^{-1})^*\omega = \omega \qquad \text{for all } n \in N.$$

The main tool we use to build this complex is again a polynomial crystallographic action. We work out an explicit formula for the invariant tuples of polynomial differential forms, and employ it in an algorithm computing the cohomology of a $\mathcal{T}$-group with coefficients in a finite dimensional real vector space with any module structure.

Cohomology of free abelian extensions

Where for $\mathcal{T}$-groups, Nomizu's theorem turned out to be an essential tool, no such result is available for free abelian extensions of $\mathcal{T}$-groups, and a different line of attack is in order. On a theoretical level, we study the relation between the cohomology of a group and of its low rank free abelian extensions, with coefficients in any module. Based on an in-depth analysis of the cohomology of a rank one free abelian extension in terms of the cohomology of the group we extend, we introduce an iterative plan of attack to relate the cohomology of free abelian extensions to the cohomology of the group we extend. The fundamental type of extension in this approach, a rank one free abelian extension, is handled by the following

Theorem. *Let G be an infinite cyclic extension of N,*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \langle \bar{t} \rangle \longrightarrow 0,$$

and M a G-module. Then, for any $p \geq 1$, there is an exact sequence

$$0 \longrightarrow \left[H^{p-1}(N, M)\right]_{\langle \bar{t} \rangle} \longrightarrow H^p(G, M) \longrightarrow \left[H^p(N, M)\right]^{\langle \bar{t} \rangle} \longrightarrow 0.$$

We then tackle the problems involved in the first iteration step, and describe the connection between the cohomology of a group and of its rank two free abelian extensions. In terms of a morphism δ^p arising as a connecting homomorphism out of the two consecutive rank one extensions, we find

Theorem. *Let G be an extension of a group N by a rank 2 free abelian group,*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \langle \bar{t}_1, \bar{t}_2 \rangle \longrightarrow 0,$$

and M a finite dimensional vector space with any G-module structure. For any $p \geq 2$, the cohomology space $H^p(G, M)$ of G with coefficients in M is isomorphic to

$$H^{p-2}(N, M)_{\langle \bar{t}_1, \bar{t}_2 \rangle} / Im\ \delta^{p-1} \oplus H^{p-1}(N, M)^{\langle \bar{t}_1, \bar{t}_2 \rangle} \oplus H^{p-1}(N, M)_{\langle \bar{t}_1, \bar{t}_2 \rangle} \oplus ker\ \delta^p.$$

Filling in the abstract relations in terms of the cohomology computation methods for $\mathcal{T}$-groups, we obtain an algorithm computating the cohomology of rank 1 and rank 2 free abelian extensions of $\mathcal{T}$-groups. Again, polynomial crystallographic actions play a crucial role. As an application we present explicit formulas for the ranks of the cohomology spaces of $\mathbb{Z}^n$-by-$\mathbb{Z}$ and split $\mathbb{Z}^n$-by-$\mathbb{Z}^2$ groups.

Cohomology of finite extensions

As for free abelian extensions, we first of all work out the relation between the cohomology of a group and of its finite or (infinite cyclic)-by-finite extensions, provided the coefficient module is uniquely divisible by the order of the finite quotient. For the finite extension case we first of all prove

Proposition. *Let E be an extension of a group N by a finite group,*

$$1 \longrightarrow N \longrightarrow E \longrightarrow F \longrightarrow 1,$$

and M an E-module uniquely divisible by the order of F. Then $H^*(E, M)$ is isomorphic to the invariants under the action of F on the cohomology $H^*(N, M)$ of N with coefficients in M.

A spectral sequence argument then shows that the cohomology of a group and its (infinite cyclic)-by-finite extensions are related as described in

Theorem. *Let G be an infinite cyclic extension of a group N,*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \langle \bar{t} \rangle \longrightarrow 0,$$

and E an extension of G by a finite group,

$$1 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0,$$

such that N is normal in E. Suppose M is an E-module uniquely divisible by the order of F. For any $p \geq 0$, there is an exact sequence

$$0 \longrightarrow \left[H^{p-1}(N, M)_{\langle \bar{t} \rangle}\right]^F \longrightarrow H^p(E, M) \longrightarrow \left[H^p(N, M)^{\langle \bar{t} \rangle}\right]^F \longrightarrow 0.$$

We use a polynomial crystallographic action of a virtually $\mathcal{T}$-group or a virtually $\mathcal{T}$-by-$\mathbb{Z}$ group to work out an algorithmic version of these relations in cohomology, based on the cohomology computation methods for $\mathcal{T}$-groups. As an application we present explicit formulas for the ranks of the cohomology spaces of a virtually nilpotent group with coefficients in any finite dimensional real vector space with an almost-trivial module structure. In particular, we obtain a formula for the Betti numbers of a compact flat Riemannian manifold in terms of the holonomy representation.

References

[1] Fried, D. Goldman, W. and Hirsch, M. *Affine manifolds with nilpotent holonomy.* Comm. math. Helv., vol. 56, 1981, pp. 487–523.

[2] Lambe, L. A. *Algorithms for computing the cohomology of nilpotent groups*, in: *Computers in geometry and topology.* Lecture Notes in Pure and Applied Mathematics, vol. 114, 1989, pp. 189–209.

[3] Nomizu, K. *On the cohomology of compact homogeneous spaces of nilpotent Lie groups.* Annals of Mathematics, vol. 59, 1954, pp. 531–538.