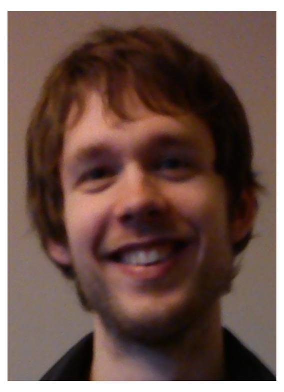
Equivariant and non-equivariant uniform embeddings into products and Hilbert spaces



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Background

Crystallographic groups, i.e. groups acting faithfully, isometrically and crystallographically on Euclidean space $\mathbb{R}^n$, occur in many places in mathematics. There are three main theorems, called the Bieberbach theorems, that describe the structure and rigidity of this class of groups. Having such nice results in this context, people studied generalizations of this setting. First, all the Bieberbach theorems were generalized to faithful, isometric and crystallographic actions on simply connected, connected, nilpotent Lie groups, equipped with a left invariant Riemannian metric.

Generalizing Bieberbach

The first part of my Ph.D. thesis [2] proves the following generalization of the first Bieberbach theorem.

Theorem 1.1. *Let M be a closed connected Riemannian manifold and let N be a simply connected, connected, nilpotent Lie group equipped with a left-invariant metric. If Γ is a group acting faithfully, properly discontinuously, cocompactly and isometrically on $M \times N$ (equipped with the product metric), then Γ contains a finite index subgroup isomorphic to a uniform lattice of N.*

The proof follows easily if one can show that the isometries on $M \times N$ split, i.e. if $\mathrm{Iso}(M \times N) = \mathrm{Iso}(M) \times \mathrm{Iso}(N)$.

Fni maps and splitting of isometries

Let M^n and N be Riemannian manifolds where M is closed. A diffeomorphism $f : M \times N \to M \times N$ is called fiberwise volume non-increasing (fni) if $\forall z \in N : \mathrm{Vol}(f(M \times \{z\})) \leq \mathrm{Vol}(M)$, where Vol refers to the standard Riemannian volume.

Theorem 1.2 (Slice theorem)**.** *If $H^n(M \times N; \mathbb{Z}_2) = \mathbb{Z}_2$ then every fni map on the product maps M-fibers $M \times \{z\}$ to M-fibers. Hence, the set of fiberwise volume non-increasing maps is a group, denoted FNIC($M \times N$).*

Theorem 1.3 (Structure theorem)**.** *We have the following split short exact sequence:*

$$1 \to K \hookrightarrow \mathit{FNIC}(M \times N) \xrightarrow{\psi} \mathit{Diffeo}(N) \to 1$$

where ψ is an explicitly known map and where $K \cong \{f : N \to \mathit{Diffeo}(M) \mid f \text{ is Fréchet differentiable}\}$.

Using the Slice theorem, we show:

Theorem 1.4 (Splitting theorem)**.** *If $H^n(M \times N; \mathbb{Z}_2) = \mathbb{Z}_2$, then the isometries of $M \times N$ split.*

Generalizing Bieberbach

A second way to generalize the first Bieberbach theorem consists of replacing Euclidean space by the class of real Hilbert spaces. We obtain the class of groups with the Haagerup property, i.e. locally compact, second countable groups that admit a metrically proper isometric action on a real Hilbert space [1].

In my thesis, I study equivariant Hilbert space compression, a notion which quantifies the speed at which orbit maps tend to infinity and thus which in some sense quantifies the degree to which a group satisfies the Haagerup property.

Definitions and standard facts I

Definition 1.5. *Let $f : G \to \mathcal{H}$ be a large-scale Lipschitz map, where G is finitely generated and equipped with the word length metric relative to a finite symmetric generating subset. The compression $R(f)$ of f is the supremum of $\alpha \in [0,1]$ such that there exist $C > 0, D \geq 0$ satisfying that $\forall x, y \in G$:*

$$(1/C)d(x,y)^\alpha - D \leq d(f(x), f(y)).$$

The equivariant Hilbert space compression of G is the $\sup\{R(f) \mid f$ is an orbit map of G under some affine isometric action of G on a Hilbert space $\}$.

Quantification idea: Equivariant Hilbert space compression quantifies how "strongly" a group satisfies the Haagerup property

Definitions and standard facts II

Definition 1.6. *The Hilbert space compression of G is the $\sup\{R(f) \mid f$ is any large-scale Lipschitz maps of G into some Hilbert space$\}$.*

Quantification idea: Hilbert space compression quantifies the degree to which a group embeds quasi-isometrically into a Hilbert space.

Nice results in this context are e.g. due to Guentner and Kaminker [3]:

Theorem 1.7. *If the compression of G is strictly greater than $1/2$, then G has Yu's property (A). If the equivariant compression of G is strictly greater than $1/2$, then G is amenable.*

We have studied the behaviour of equivariant and non-equivariant compression under group constructions.

Free products and HNN-extensions

Theorem 1.8. *Denote the Hilbert space compressions of finitely generated groups G_1 and G_2 by α_1 and α_2 respectively. Let F be a common finite subgroup. The Hilbert space compression α of the amalgamated free product $G = G_1 *_F G_2$ satisfies*

$$\min(\alpha_1, \alpha_2, 1/2) \leq \alpha \leq \min(\alpha_1, \alpha_2).$$

Theorem 1.9. *Consider $G := \mathit{HNN}(H, F, \theta)$ where both F and $\theta(F)$ are finite index subgroups of the finitely generated group H. Equip H with the induced metric d_{in} from G. Denoting the Hilbert space compressions of (H, d_{in}) and G by α_1 and α respectively, we get*

$$\alpha_1/3 \leq \alpha \leq \alpha_1.$$

Group extensions

Theorem 1.10. *Assume that Γ is a finitely generated group that fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G has polynomial growth and if H with the induced metric from Γ has compression α, then the compression of Γ is at least $\alpha/3$.

Theorem 1.11. *Assume that Γ is a finitely generated group that fits in a short exact sequence*

$$1 \to H \to \Gamma \xrightarrow{\pi} G \to 1.$$

If G is a hyperbolic group in the sense of Gromov and if H, with the induced metric from Γ, has Hilbert space compression α, then the Hilbert space compression of Γ is at least $\alpha/5$.

Free products (Equivariant case)

Theorem 1.12. *Let G_1 and G_2 be finitely generated groups with equivariant Hilbert space compressions equal to α_1 and α_2 respectively. Denote $G = G_1 *_F G_2$ an amalgamated free product where F is a finite subgroup of both G_1 and G_2. If α denotes the equivariant Hilbert space compression of G, then*

1. $\alpha = 1$ if F is of index 2 in both G_1 and G_2,
2. $\alpha = \alpha_1$ if $F = G_2$ and $\alpha = \alpha_2$ if $F = G_1$,
3. $\alpha = \min(\alpha_1, \alpha_2, 1/2)$ otherwise.

HNN-extensions (Equivariant case)

Theorem 1.13. *Let H be a finitely generated group with equivariant Hilbert space compression α_1 and assume that $F < H$ is finite. Denoting the equivariant Hilbert space compression of HNN(H, F, θ) by α, we get*

1. $\alpha = 1$ whenever $F = H$,
2. $\alpha = \min(\alpha_1, 1/2)$ otherwise.

References

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[3] E.Guentner, J.Kaminker, 'Exactness and uniform embeddability of discrete groups', *Journal of the London Mathematical Society* 70, no.3 (2004), 703–718